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本科毕业论文

论文题目：A More Advanced Model for GW Tests of the Kerr Metric with Binary Black Holes

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Contents

List of Figures

Abstract

Gravitational wave observations of binary black holes provide a suitable arena to test the fundamental properties of gravity in the strong-field regime. We intend to construct a more accurate theoretical gravitational wave model to include black hole spins to test the Kerr hypothesis. After reviews on the post-Newtonian formalism and the effective one-body formalism, we discuss the application of these methods for the calculations of effective spin and figure out the exact calculation process. Finally, we outline the further calculation process to obtain the modified constraints on KRZ metric.

Keywords: gravitational waves; post-Newtonian theory; effective one body theory; effective spin; KRZ metric

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List of Symbols

c	the speed of light
v	velocity
$\mathbf{P}, \mathbf{p}$	momentum
ε	the dimensionless parameter to characterize PN order
U	gravitational potential
p	pressure
ρ	density
$\mathbf{S}$	classical spin
$g_{\mu\nu}$	the space-time metric
g	the metric determinant
$(\mu, \nu, \rho, ...)$	subscript representing both time and position
$(i, j, k, ...)$	subscript representing position
S	action
τ	proper time
t	coordinate time
G	the gravitational constant
$\Gamma^{\lambda}_{\mu\nu}$	Christoffel symbols
$t^{\alpha\beta}_{LL}$	Landau-Lifshitz pseudotensor
$\eta_{\mu\nu}$	the Minkowski metric
$\triangle$	Laplacian operator

Chapter 1
Introduction

The general theory of relativity, since its proposition over a century ago, has been applied to a variety of astrophysical phenomena in our Universe. While it has been successfully tested in the weak-fields, only recently the strong-field predictions of Einstein's gravity have become testable in a variety of ways.[1].

Among all the astrophysical objects, black hole is a perfect candidate to study strong-field gravity. In general relativity, the solution to Einstein's field equation of the black hole is the Kerr metric, in which the black hole is characterized only by its mass and spin[2]. The assumption that astrophysical black holes are described by the Kerr metric is known as the Kerr hypothesis[3]. We can test the Kerr hypothesis to test feneral relativity. By introducing additional parameters, people can obtain the parameterized metrics deforming the black hole away from the Kerr solution. Through measuring these parameters and comparing them with the Kerr metric, the validity of general relativity can then be confirmed if the parameterized metric converges to the Kerr metric in measurement. In this sense, accurate models and high-quality data are the key to the test.

In this thesis, we introduce the basic knowledge one needs to study gravitational waves and propose a strategy to include spin effect into the modeling of gravitational waves, which has not been done before. We adopt the KRZ metric[4] as the parameterized deforming metric.

The contents are arranged as follows. Chapter 2 reviews the post-Newtonian (PN) and effective one-body (EOB) formalism which is useful in gravitational wave analysis. Chapter 3 summarizes previous works on effective spin calculations and figures out the exact calculation process of the effective spin. Chapter 4 introduces the KRZ metric and describes the further calculation strategy. Chapter 5 has a discussion on the problems in working toward the effective deformation parameters, which is in our initial envision.

Chapter 2

Review on PN and EOB Formalism

2.1 Post-Newtonian Formalism

The post-Newtonian formalism is an approximation method to solve the Einstein's field equation. It focuses on solving the problem of an isolated and self-gravitating system, where the objects inside are slowly moving and weakly pressed. The domain of its validity is limited to the near zone of the source - the region surrounding the source that is of small extent with respect to the wavelength of the gravitational waves.

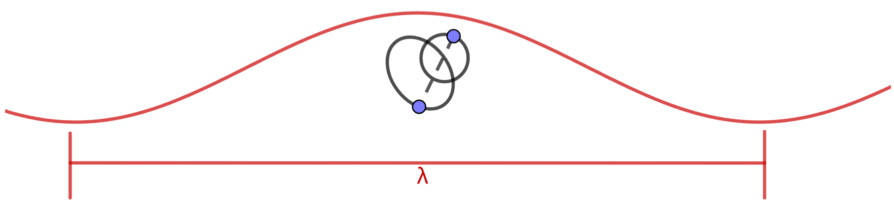

Figure 2-1 relative size of the source and gravitational waves

The post-Newtonian approximation has been developed from the early days of general relativity[5]. It successfully gives answers to the problems of motion and gravitational radiation of systems of compact objects[6]. Three crucial applications are:

1. The motion of N point-like objects at the first post-Newtonian approximation level is taken into account to describe the solar system dynamics;

2. The gravitational radiation reaction force has been experimentally verified by the observation of the secular acceleration of the orbital motion of the Hulse-Taylor binary pulsar PSR 1913+16;

3. The analysis of gravitational waves emitted by inspiralling compact binaries needs the prior results of the equation of motion and radiation field up to very high post-Newtonian order.

Since post-Newtonian approximation is only valid in the near zone, it is as a consequence that the post-Newtonian expansion(nonlinear 1/c-expansion) cannot incorporate the boundary conditions at infinity, which determine the radiation reaction force in the source's local equation of motion.

The accuracy of the post-Newtonian approximation is often described in terms of the post-Newtonian order, whose abbreviation is the PN order. Its characteristic parameter is ε which reads

$$\varepsilon \sim \frac{U}{c^2} \sim \frac{v^2}{c^2} \sim \frac{p}{\rho c^2} \tag{2.1}$$

where U is the gravity potential, v is the particle velocity inside the source, p is pressure and ρ is the matter density.

When we refer to the action S at nPN order, we are required to evaluate the metric coefficients to the following orders of approximation

$$\begin{aligned} g_{00} &\sim O(\varepsilon^{n+1}) \\ g_{0i} &\sim O(\varepsilon^{n+\frac{1}{2}}) \\ g_{ij} &\sim O(\varepsilon) \end{aligned} \tag{2.2}$$

The order needs to descend because of the additional factors of v/c in the action

$$\begin{aligned} S &= -mc^2 \int_1^2 d\tau = -mc \int_1^2 \sqrt{-g_{\alpha\beta} \frac{dr^\alpha}{dt} \frac{dr^\beta}{dt}} dt \\ &= -mc^2 \int_1^2 \sqrt{-g_{00} - 2g_{0j} \frac{v^j}{c} - g_{ij} \frac{v^i v^j}{c^2}} dt \end{aligned} \tag{2.3}$$

where the action functional is the elapsed proper time $\int_1^2 d\tau$ along a parameterized curve $r^\alpha(\tau)$ linking the fixed events 1 and 2.

This section is a rough sketch of the post-Newtonian theory. For more details, see references[6] [7] [8].

2.1.1 Einstein Field Equation

The dynamics of the gravitational field is described by the Einstein field equation, which can be obtained by varying the action S with respect to the spacetime metric $g_{\mu\nu}$,

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R + S_m[\Psi, g_{\mu\nu}] \tag{2.4}$$

The first term in S is the Einstein-Hilbert action of the spacetime curvature. $d^4x\sqrt{-g}$ is the invariant scalar volume, $R = g^{\mu\nu} R_{\mu\nu}$ is the curvature scalar with $R_{\mu\nu}$ the Ricci tensor. Notice that Rienmann tensor and Ricci tensor are defined respectively as follows,

$$R^\lambda_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu} \Gamma^\lambda_{\sigma\rho} \tag{2.5}$$

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{2.6}$$

The second term in S_{EH} is the action of the matter source with Ψ representing the matter field.

When the metric has a small change $\delta g_{\mu\nu}$, the variation of the action can be written as

$$\delta S = \delta S_m + \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} \tag{2.7}$$

where S is a redefined action by adding a boundary term to S_{EH} and $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.8}$$

After dealing with the variation of the curvature part, we now carefully consider the matter action. Since the matter action is defined as

$$S_m[\Psi, g_{\mu\nu}] = \int d^4x \mathscr{L}_m(\Psi, \partial\Psi, g_{\mu\nu}), \tag{2.9}$$

we can get the variation of S_m with respect to $g_{\mu\nu}$, which is

$$\delta S_m = \int d^4x \frac{\partial \mathscr{L}_m}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \tag{2.10}$$

Thus we can define the stress-energy of the matter field

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathscr{L}_m}{\partial g_{\mu\nu}}. \tag{2.11}$$

Due to the least action principle, we can impose the condition $\delta S = 0$ to equation (2.7) and combine it with equation (2.10). Finally, we acquire the famous Einstein field equation

$$G^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}, \tag{2.12}$$

which is the equation of motion of matter fields.

2.1.2 Landau-Lifshitz Formulation

After constructing the Einstein field equation, we will move to rewrite it in a explicitly solvable form. Therefore we introduce the Landau-Lifshitz formulation of the field equation, transforming it to a formal wave equation.

In this framework, the main variable is the gothic inverse metric density

$$\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}. \tag{2.13}$$

Knowledge of the gothic metric is sufficient to determine the metric itself due to the relation

$$det(\mathfrak{g}^{\mu\nu}) = g. \tag{2.14}$$

Then we define a new tensor density $H^{\alpha\mu\beta\nu}$ to build the left-hand side of the field equation.

$$H^{\alpha\mu\beta\nu} \equiv \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \tag{2.15}$$

It turns out that $H^{\alpha\mu\beta\nu}$ satisfy the identity containing its partial derivative

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -g(2G^{\alpha\beta} + \frac{16\pi G}{c^4} t^{\alpha\beta}_{LL}) \tag{2.16}$$

where $G^{\alpha\beta}$ is the Einstein tensor and

$$\begin{aligned}(-g)t^{\alpha\beta}_{LL} \equiv & \frac{c^4}{16\pi G}[\partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\ & - g^{\alpha\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu} g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma}]\end{aligned} \tag{2.17}$$

defines the Landau-Lifshitz pseudotensor, which represents the distribution of gravitational-field energy. Substituting the Einstein's equation (2.12) into equation (2.15), we obtain

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL}) \tag{2.18}$$

Because of the antisymmetry of $H^{\alpha\mu\beta\nu}$ in the last pair of indices, we have the following trivial identity

$$\partial_{\beta\mu\nu} H^{\alpha\mu\beta\nu} = 0. \tag{2.19}$$

Combining this identity with equation (2.18), the conservation equations for the total energy-momentum pseudotensor come out as a consequence.

$$\partial_\beta[-g(T^{\alpha\beta} + t^{\alpha\beta}_{LL})] = 0 \tag{2.20}$$

These equations are equivalent to the usual expression

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{2.21}$$

This is an exact reformulation of the standard form of the theory with no approximation involved and no restriction imposed on the choice of coordinates.

In the next step, we will choose a specific gauge by adding the harmonic gauge conditions

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0. \tag{2.22}$$

After introducing the potentials

$$h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}, \tag{2.23}$$

these gauge conditions become

$$\partial_\beta h^{\alpha\beta} = 0. \tag{2.24}$$

The introduction of the potentials $h^{\alpha\beta}$ and the harmonic gauge conditions largely simplifies the appearance of the Einstein field equations. It is easy to verify that

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta} - \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu}, \tag{2.25}$$

where the $\Box$ is the flat spacetime wave operator

$$\Box = \eta^{\mu\nu}\partial_{\mu\nu}. \tag{2.26}$$

Plugging (2.25) into (2.18) and keeping only the wave operator on the left-hand side, we get the formal wave equation for the potentials $h^{\alpha\beta}$

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{2.27}$$

where $\tau^{\alpha\beta}$ is the effective energy-momentum pseudotensor for the wave equation.

$$\tau^{\alpha\beta} \equiv -g(T^{\alpha\beta} + t^{\alpha\beta}_{LL} + t^{\alpha\beta}_{H}) \tag{2.28}$$

$$(-g)t^{\alpha\beta}_H \equiv \frac{c^4}{16\pi G}(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta}) \tag{2.29}$$

2.1.3 Solution of the Wave Equation

Through previous work, we obtain the relaxed Einstein field equation (2.27), which is independent by itself without imposing any gauge conditions. This is a formal wave equation with a source. So we can immediately write its formal solution out

$$h^{\alpha\beta}(t,\mathbf{x}) = \frac{4G}{c^4}\int d^3x' \frac{\tau^{\alpha\beta}(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|}. \tag{2.30}$$

The domain of integration extends over the past light cone $\mathscr{C}(x)$ of the field point $x = (ct,\mathbf{x})$. Notice that x' is the matter source point.

To analyse the solution structure, we partition the integration domain into two parts - the near-zone domain $\mathscr{N}(x)$ and the wave zone domain $\mathscr{W}(x)$.Therefore we can write

$$h^{\alpha\beta}(x) = h^{\alpha\beta}_{\mathscr{N}}(x) + h^{\alpha\beta}_{\mathscr{W}}(x) \tag{2.31}$$

The magnitude of the boundary radius R of the near zone and the wave zone is of the same order as the characteristic wavelength λ_c of the radiation emitted by the source. For slowly moving source, λ_c is way larger than the characteristic size a of the material source, as is shown in Fig.2-1. Thus, the potential behaves very differently in the near zone and the wave zone as a result of the comparison of the retarded time variable.

In spacetime, the sphere of radius R traces a near-zone world tube $\mathscr{D}$. Then the near zone $\mathscr{N}(x)$ is the intersection between $\mathscr{C}(x)$ and $\mathscr{D}$. And the wave zone is $\mathscr{C}(x)$-$\mathscr{N}(x)$. The specific integration domains are shown in Fig.2-2.

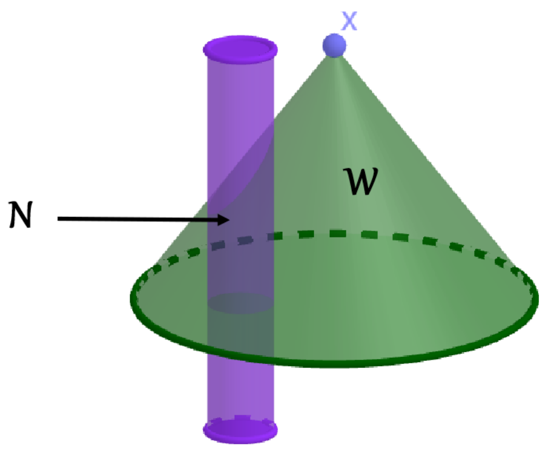

Figure 2-2 integration domains for the retarded solution of the wave equation

Because of the nonlinearity of gravity, $t^{\alpha\beta}_{LL}$ and $t^{\alpha\beta}_{H}$ are not confined to the near zone. Special integrals must be used to perform the integrals over $\mathscr{W}$, but for most physical systems of interest the wave-zone integrals generate higher order corrections to the dominant terms, which comes from the near-zone integrals. So we now only care about the term $h^{\alpha\beta}_{\mathscr{N}}(x)$.

When the field point is situated in the wave zone, we figure out the Taylor expansion of the variable $r = |\mathbf{x} - \mathbf{x}'|$ in $h^{\alpha\beta}_{\mathbf{N}}(x)$ about $\mathbf{x}' = \mathbf{0}$ since $\mathbf{x}'$ lies within the near zone and $r \gg R$. The result is

$$h^{\alpha\beta}_{\mathscr{N}}(t, \mathbf{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L [\frac{1}{r} \int_{\mathscr{M}} \tau^{\alpha\beta}(t - \frac{r}{c}, \mathbf{x}') x'^L d^3x'], \tag{2.32}$$

where the integration domain $\mathscr{M}$ is a surface of constant time equal to the retarded-time variable $t_r = t - r/c$. Notice that L is a multi-index notation and x^L stands for $x^{j_1 j_2 j_3 \cdots j_l} = x^{j_1} x^{j_2} x^{j_3} \cdots x^{j_l}$. It is the same with notation ∂_L. In $r \to \infty$ limit, Equation (2.32) is approximated to

$$\begin{aligned} h^{\alpha\beta}_{\mathscr{N}}(t, \mathbf{x}) &= \frac{4G}{c^4 r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_{\mathscr{M}} \partial_L \tau^{\alpha\beta}(t - \frac{r}{c}, \mathbf{x}') x'^L d^3x' + O(\frac{1}{r^2}) \\ &= \frac{4G}{c^4 r} \sum_{l=0}^{\infty} \frac{1}{l! c^l} n_L \frac{d^l}{dt_r^l} \int_{\mathscr{M}} \tau^{\alpha\beta}(t_r = t - \frac{r}{c}, \mathbf{x}') x'^L d^3x' + O(\frac{1}{r^2}). \end{aligned} \tag{2.33}$$

Next we evaluate $h^{\alpha\beta}_{\mathscr{N}}(x)$ when x is situated in the near zone. In this sense, both $\mathbf{x}$ and $\mathbf{x}'$ lie in the near zone, for which $r = |\mathbf{x} - \mathbf{x}'|$ can be treated as a small quantity. In this situation, the difference between the retarded time t_r and the time of the field point t is so small that we can deem them as the same. So after Taylor expanding the time-dependence of the source, we get the result

$$h^{\alpha\beta}_{\mathscr{N}}(t, \mathbf{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l! c^l} (\frac{\partial}{\partial t})^l \int_{\mathscr{M}} \tau^{\alpha\beta}(t, \mathbf{x}') r^{l-1} d^3x', \tag{2.34}$$

where the domain of integration $\mathscr{M}$ is a surface of constant time t.

Above all is the formal solution of the wave equation (2.27). We should notice that the equation is not solved actually because $h^{\alpha\beta}$ appears in both sides. In order to construct solutions for a particular choice of matter variables, we proceed by iterations-plugging the result of $h^{\alpha\beta}$ in the last step into the integral in the right-hand side, solving the new wave equation to get new result of $h^{\alpha\beta}$ for the next iteration. In the zeroth iteration, we set the initial $h^{\alpha\beta}_0$ to be 0. One continues iterating until the desired precision is reached.

2.1.4 Compact Binary Systems

Before considering the compact binary system, we first introduce a general situation in which the matter source is composed of N point-like particles.

The matter action can be written as

$$\begin{aligned} S_m &= \sum_A -m_A \int d\tau_A \\ &= \sum_A -m_A \int dt \sqrt{-(g_{\mu\nu})_A v^\mu_A v^\nu_A}, \end{aligned} \tag{2.35}$$

where A denotes the N particles. Substituting this matter action into formula (2.10) and (2.11), we get the specific form of the energy-momentum tensor of this source. Then the

Einstein field equation is constructed. The Fokker action is obtained by inserting back into an explicit PN iterated solution of the field equations (2.27), given as an explicit PN metric $\overline{g}_{\mu\nu}(\mathbf{x};\mathbf{y}_B,\mathbf{v}_B,...)$[9].

$$\begin{aligned} S_F[\mathbf{y}_B(t),\mathbf{v}_B(t),...] = &\int dt \int d^3\mathbf{x}\mathscr{L}[\overline{g}_{\mu\nu}(\mathbf{x};\mathbf{y}_B(t),\mathbf{v}_B(t),...)] \\ &- \sum_A m_A \int dt \sqrt{-\overline{g}_{\mu\nu}(\mathbf{y}_A;\mathbf{y}_B,\mathbf{v}_B,...)v_A^\mu v_A^\nu} \end{aligned} \tag{2.36}$$

Then when $N = 2$, the system reduces to a compact binary system. And we can use the Fokker action to derive the corresponding equation of motions and other physical quantities.

2.1.5 Gravitational Radiation Reaction

Gravitational radiation reaction is the reason for the loss of energy, momentum and angular momentum in the original matter source dynamical system. There is an additional radiation reaction force $\mathbf{F}_{reac}$ to the post-Newtonian Euler equation of hydrodynamics.

$$\rho\frac{dv^i}{dt} = -\partial^i p + \rho\partial^i U + F^i_{reac} \tag{2.37}$$

Notice that U is the gravitational potential in Poisson's equation and the radiation reaction force starts at the 2.5PN order and is related to the gravitational radiation.

Before we talk about the radiation reaction dissipative effect, we first neglect this effect and illustrate the conservative dynamics. We focus on the compact binary system. In the PN solution process, we define a scalar potential V:

$$\Box V = -4\pi G\sigma, \tag{2.38}$$

$$\sigma = \frac{T^{00}+T^{ii}}{c^2}. \tag{2.39}$$

The formal solution of this equation reads

$$V(t,\mathbf{x}) = \int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'), \tag{2.40}$$

which is a retarded potential. If we only keep the conservative part, we can write it as follows,

$$V = (\triangle^{-1}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\triangle^{-2}+\frac{1}{c^4}\frac{\partial^4}{\partial t^4}\triangle^{-3}+...)[-4\pi G(\tilde{\mu}_1\delta(\mathbf{x}-\mathbf{y}_1)+\tilde{\mu}_2\delta(\mathbf{x}-\mathbf{y}_2))] \tag{2.41}$$

$$\tilde{\mu}_A = \frac{m_A(1+\frac{v_A^2}{c^2})}{\sqrt{(g)_A(g_{\mu\nu})_A v_A^\mu v_A^\nu}}. \tag{2.42}$$

Now we consider the radiation part. We should notice that there is a retarded time variable in the scalar potential in Equation (2.40). So when we calculate the expansion of the iteration from the retarded integral, we obtain

$$\begin{aligned} V(t,\mathbf{x}) = \triangle^{-1}(-4\pi\sigma) + \frac{1}{c}\frac{d}{dt}\int d^3x'\sigma(t,\mathbf{x}') + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\triangle^{-2}(-4\pi\sigma) \\ + \frac{1}{c^3}\frac{d^2}{dt^2}\int d^3x'|\mathbf{x}-\mathbf{x}'|^2\sigma(t,\mathbf{x}') + ... \end{aligned} \tag{2.43}$$

Subtracting the conservative parts which are all the even-PN terms and inserting the mass conservation condition which cancels the second term, we finally obtain the leading radiation reaction term, the fourth term. We denote this term as

$$V_{reac}(t,\mathbf{x}) = -\frac{G}{5c^5}x^ix^jQ^{(5)}_{ij}(t) + O(\frac{1}{c^7}). \tag{2.44}$$

The quadrupole moment in it is defined by

$$Q_{ij}(t) \equiv \int d^3\mathbf{x}'\rho(x')(x'_ix'_j - \frac{1}{3}\delta_{ij}\mathbf{x}'^2), \tag{2.45}$$

and the up-left script (n) denotes the nth order of its time derivative.

Then we can evaluate the radiation reaction force by

$$\begin{aligned} F^{reac}_i &= \rho\partial_i V^{reac} \\ &= -\frac{2G}{5c^5}\rho x^j Q^{(5)}_{ij}(t) + O(\frac{1}{c^7}). \end{aligned} \tag{2.46}$$

Applying the energy balance equation and plugging in the Equation(2.46), we get

$$\begin{aligned} \frac{dE}{dt} &= \int d^3xv^iF^{reac}_i \\ &= -\frac{2G}{5c^5}Q^{(5)}_{ij}(t)\int d^3x\rho x^jv^i \\ &= -\frac{G}{5c^5}Q^{(5)}_{ij}(t)Q^{(1)}_{ij}(t) \\ &= -\frac{G}{5c^5}Q^{(3)}_{ij}Q^{(3)}_{ij} + \frac{df}{dt}, \end{aligned} \tag{2.47}$$

$$f = \frac{G}{5c^5}(Q^{(1)}_{ij}Q^{(4)}_{ij} - Q^{(2)}_{ij}Q^{(3)}_{ij}), \tag{2.48}$$

where f corresponds to the secular effect. For secular quasi-periodic sources, the time average of df/dt is negligible, compared to the energy flux. So we time-average the last line in Equation(2.47) and ignore the second term to get the famous quadrapole formula

$$\left\langle\frac{dE}{dt}\right\rangle = -\frac{G}{5c^5}\left\langle Q^{(3)}_{ij}Q^{(3)}_{ij}\right\rangle, \tag{2.49}$$

where $\langle F\rangle$ denotes the average of function F over time.

2.2 Effective One Body Formalism

In this part we will briefly outline the main features and processes of the effective one-body approach for two body problems. For more technical details, see reference[10] [11] [12].

The main idea of this approach is to transform the two-body dynamics into an effective one-body system in which we only need to consider the motion of a test particle in some effective external metric. If we do not count in the radiation reaction effect, the effective metric will be a static and spherically symmetric deformation of the Schwarzschild metric.

When the two objects have comparable masses m_1, m_2, the one-body dynamics of them will be one particle of reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ moving in some effective metric generated by the total mass $M = m_1 + m_2$,

$$ds^2_{eff} = -A(R_{eff})c^2 dt^2_{eff} + \frac{D(R_{eff})}{A(R_{eff})} dR^2_{eff} + R^2_{eff}(d\theta^2_{eff} + sin^2\theta_{eff} d\phi^2_{eff}) \quad (2.50)$$

where

$$\begin{aligned} A(R) &= 1 + \frac{a_1}{c^2 R} + \frac{a_2}{c^4} + \frac{a_3}{c^6 R^3}, \\ D(R) &= 1 + \frac{d_1}{c^2 R} + \frac{d_2}{c^4 R^2}. \end{aligned} \quad (2.51)$$

We use the explicit, post-Newtonian expanded classical equations of motion of a gravitationally interacting system of two compact objects in harmonic coordinates, which have the form

$$\mathbf{a}_a = \mathbf{A}^{con}_a(\mathbf{z}_b, \mathbf{v}_b) + \mathbf{A}^{reac}_a(\mathbf{z}_b, \mathbf{v}_b), \quad (2.52)$$

where $\mathbf{A}^{con}$ denotes the time-symmetric part of the equations of motion, which means it is conservative. And $\mathbf{A}^{reac}$ is the time-antisymmetric part, corresponding to the radiation reaction effects. They can all be post-Newtonian expanded. $\mathbf{z}_a$, $\mathbf{v}_a$, $\mathbf{a}_a$ represents the positions, velocities and accelerations of the two bodies in harmonic coordinates.

We first focus on the conservative part $\mathbf{A}^{con}$. Since it has been explicitly shown that the 3PN dynamics is Poincaré invariant[13], we can go to the center of mass frame and transform the two-body problem to an effective one. After a coordinate transformation from harmonic coordinates $\mathbf{z}_a$ to ADM coordianates $\mathbf{q}_a$[14], the dynamics of the relative coordinates $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ is defined by a 3PN Hamiltonian $H(\mathbf{q}, \mathbf{p})$. Then we use the unique effective metric defined in Equation(2.50) to get the effctive Hamiltonian

$$H_{eff}(\mathbf{q}_{eff}, \mathbf{p}_{eff}) = -\int \mu c \, ds_{eff}. \quad (2.53)$$

After this, we perform a canonical transformation $(\mathbf{q}_{eff}, \mathbf{p}_{eff}) \to (\mathbf{q}, \mathbf{p})$ and an energy transformation $H = f(H_{eff})$ corresponding to an energy-dependent canonical rescaling of the time coordinate

$$\frac{dt_{eff}}{dt} = \frac{dH}{dH_{eff}}. \quad (2.54)$$

Notice that for up-to 3PN Hamiltonian, the energy mapping relation would be[15]

$$\frac{H_{eff}}{\mu c^2} = \frac{H^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}. \quad (2.55)$$

Finally the effective Hamiltonian can be mapped onto the real Hamiltonian and we can figure out the parameters in the defination of the effective metric.

We can include the radiation reaction effect at 2.5PN order which is well-defined, although till now the separation of the dynamics in a conservative part plus a reactive part has not been proved at higher PN orders. By adding a reactive Hamiltonian H_{reac} to the previous Hamiltonian, the real Hamiltonian can be thus improved.

$$H_{reac}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) = -h_{ij}^{TTreac}(t)[\frac{p_1^i p_1^j}{2m_1} + \frac{p_2^i p_2^j}{2m_2} - \frac{1}{2}Gm_1m_2\frac{(q_1^i - q_2^i)(q_1^j - q_2^j)}{|\mathbf{q}_1 - |\mathbf{q}_2|^3}] \quad (2.56)$$

where

$$h_{ij}^{TTreac}(t) = -\frac{4G}{5c^5}Q_{ij}^{(3)}(t) \quad (2.57)$$

and $Q_{ij}(t)$ is the quadrupole moment of the two-body system.

The effective one-body formalism is a good non-perturbative re-summation of the standard post-Newtonian-expanded results. It gives a simple way to uplift the precision of many PN-expanded results. The explicit results can be found in references[16]. The two-body system we talked above has neglected the spin of the two objects and only cared about the orbital dynamics of the system. If the spins are included, the Hamiltonian will be refined and the accuracy can also be raised. In this paper, we only focus on the leading order linear term of spin effect and leave out the non-linear term. To consider the non-linear term, see references[17] [18] [19].

Chapter 3
Calculation of Effective Spin

3.1 QFT calculation

The method of using quantum field theory (QFT) of general relativity to include spin effects into the gravitational two-body problem is derived mainly by B.M.Barker and R.F.O'Connell in references[20] [21] [22].

They considered the one-graviton exchange interaction of two particles of spin $\frac{1}{2}$, calculated the scattering matrix element in the center of mass system and transformed the Dirac spinors into the Pauli spinors. After all these operations, they successfully got the gravity potential $V(\mathbf{k})$ in momentum space from the matrix element. Then they Fourier-transformed $V(\mathbf{k})$ into the position space and took the nonrelativistic approximation to finally get

$$\begin{aligned}V(\mathbf{r}) = &-\frac{Gm_1m_2}{r}[1+(4+\frac{3m_1}{2m_2}+\frac{3m_2}{2m_1})\frac{\mathbf{P}^2}{m_1m_2c^2}] \\ &+\frac{4\pi G\hbar^2}{c^2}(1+\frac{3m_2}{8m_1}+\frac{3m_1}{8m_2})\delta(\mathbf{r}) \\ &+\frac{G\hbar^2}{4c^2r^3}[\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2] \\ &+\frac{2\pi G\hbar^2}{3c^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\delta(\mathbf{r}) \\ &+\frac{G}{c^2r^3}(1+\frac{3m_2}{4m_1})\hbar\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{P})+\frac{G}{c^2r^3}(1+\frac{3m_1}{4m_2})\hbar\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{P}).\end{aligned} \tag{3.1}$$

The last line is gravitational potentials linear to spin. Taking $\frac{1}{2}\hbar\boldsymbol{\sigma}_1 \to \mathbf{S}_1$, $\frac{1}{2}\hbar\boldsymbol{\sigma}_2 \to \mathbf{S}_2$, we have

$$V_{S_1} = \frac{2G}{c^2r^3}(1+\frac{3m_2}{4m_1})\mathbf{S}_1\cdot(\mathbf{r}\times\mathbf{P}), \tag{3.2}$$

$$V_{S_2} = \frac{2G}{c^2r^3}(1+\frac{3m_1}{4m_2})\mathbf{S}_2\cdot(\mathbf{r}\times\mathbf{P}). \tag{3.3}$$

Thus, the spin-dependent terms in the center of mass frame at the lowest PN order have been explicitly derived,

$$\begin{aligned}H_S^{PN}(\mathbf{x},\mathbf{P},\mathbf{S}_1,\mathbf{S}_2) &= V_{S_1}+V_{S_2}+O(\frac{1}{c^4}) \\ &= \frac{2G}{c^2r^3}[(1+\frac{3m_2}{4m_1})\mathbf{S}_1+(1+\frac{3m_1}{4m_2})\mathbf{S}_2]\cdot(\mathbf{r}\times\mathbf{P}).\end{aligned} \tag{3.4}$$

3.2 EOB calculation

After working out the linear terms in spin effect, people use the effective one-body (EOB) formalism to relate the result above to the effective spin $\mathbf{S}_{eff}$ in the effective one-body system[23].

The effective one-body dynamics at 3PN order was given by an Hamilton-Jacobi equation

$$\mu^2 + g_{eff}^{\alpha\beta} p_\alpha^{eff} p_\beta^{eff} + Q_4(p^{eff}) = 0, \tag{3.5}$$

where $g_{eff}^{\alpha\beta}$ is the effective metric and $Q_4(p^{eff})$ is the additional quartic-in-momenta contribution.

Define

$$\alpha \equiv \frac{1}{\sqrt{-g_{eff}^{00}}} \quad \beta^i \equiv \frac{g_{eff}^{0i}}{g_{eff}^{00}} \quad \gamma^{ij} \equiv g_{eff}^{ij} - \frac{g_{eff}^{0i} g_{eff}^{0j}}{g_{eff}^{00}}, \tag{3.6}$$

which is equivalent to

$$g_{eff}^{00} = -\frac{1}{\alpha^2} \quad g_{eff}^{0i} = -\frac{\beta^i}{\alpha^2} \quad g_{eff}^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2}. \tag{3.7}$$

Notice that the conserved effective energy is $E_{eff} = -p_0^{eff}$ and $p_\nu^{eff} = \frac{\partial S}{\partial x_{eff}^\nu}$.

After plugging the redefined metric parameters into Equation (3.5) and dropping the ”eff” label on position and momentum variables for convenience, it is reduced to

$$(E_{eff} - \beta^i p_i)^2 = \alpha^2[\mu^2 + \gamma^{ij} p_i p_j + Q_4(p)]. \tag{3.8}$$

Then the effective Hamiltonian reads

$$E_{eff} = H_{eff}(\mathbf{x}, \mathbf{p}, ...) = \beta^i p_i + \alpha\sqrt{\mu^2 + \gamma^{ij} p_i p_j + Q_4(p)}. \tag{3.9}$$

At the lowest PN order, the addition of a spin S_{eff} onto an initially spherical symmetric metric leads to an off-diagonal term in the metric

$$\beta^i \simeq -g^{0i} \simeq -g_{0i} \simeq \frac{2G}{r^3} \epsilon^{ijk} S_{eff}^j x^k. \tag{3.10}$$

Inserting this term in Equation (3.9) and PN-expanding it yields

$$\delta_{S_{eff}} H_{real} \simeq \beta^i p_i \simeq \frac{2G}{c^2 r^3} \epsilon^{ijk} p_i S_{eff}^j x^k = \frac{2G}{c^2 r^3} \mathbf{S}_{eff} \cdot (\mathbf{x} \times \mathbf{p}). \tag{3.11}$$

Comparing with Equation (3.4) we can obtain the specific spin transformation relation in the leading PN order

$$\mathbf{S}_{eff} = (1 + \frac{3m_2}{4m_1})\mathbf{S}_1 + (1 + \frac{3m_1}{4m_2})\mathbf{S}_2. \tag{3.12}$$

Chapter 4

Modification of the Constraints on KRZ Deformation Parameters

4.1 KRZ Metric

After obtaining the effective spin, we now take it in to account and turn to constrain the KRZ deformation parameters. The KRZ metric can be written in the following form in the Boyer-Lindquist coordinates[1] [4]:

$$ds^2 = -\frac{N^2 - W^2 sin^2\theta}{K^2}dt^2 + \frac{\Sigma B^2}{N^2}dr^2 + \Sigma r^2 d\theta^2 - 2W r sin^2\theta dt d\phi + K^2 r^2 sin^2\theta d\phi^2, \tag{4.1}$$

where

$$\begin{aligned} N^2 =& (1 - \frac{r_0}{r})[1 - \frac{\epsilon_0 r_0}{r} + (k_{00} - \epsilon_0)\frac{r_0^2}{r^2} + \frac{\delta_1 r_0^3}{r^3}] \\ &+ \{\frac{a_{20} r_0^3}{r^3} + \frac{a_{21} r_0^4}{r^4} + \frac{k_{21} r_0^3}{r^3[1 + \frac{k_{22}(1-\frac{r_0}{r})}{1+k_{23}(1-\frac{r_0}{r})}]}\}cos^2\theta, \end{aligned} \tag{4.2}$$

$$B = 1 + \frac{\delta_4 r_0^2}{r^2} + \frac{\delta_5 r_0^2}{r^2}cos^2\theta, \tag{4.3}$$

$$\Sigma = 1 + \frac{a_*^2 M^2}{r^2}cos^2\theta, \tag{4.4}$$

$$W = \frac{1}{\Sigma}(\frac{w_{00} r_0^2}{r^2} + \frac{\delta_2 r_0^3}{r^3} + \frac{\delta_3 r_0^3}{r^3}cos^2\theta), \tag{4.5}$$

$$K^2 = 1 + \frac{a_* M W}{r} + \frac{1}{\Sigma}\{\frac{k_{00} r_0^2}{r^2} + \frac{k_{21} r_0^3}{r^3[1 + \frac{k_{22}(1-\frac{r_0}{r})}{1+k_{23}(1-\frac{r_0}{r})}]}cos^2\theta\}, \tag{4.6}$$

and

$$\begin{aligned} &r_0 = M(1 + \sqrt{1 - a_*^2}), \quad \epsilon_0 = \frac{2M - r_0}{r_0}, \quad a_{21} = -\frac{a_*^4 M^4}{r_0^4} + \delta_6, \\ &k_{00} = k_{22} = k_{23} = \frac{a_*^2 M^2}{r_0^2}, \quad k_{21} = \frac{a_*^4 M^4}{r_0^4} - \frac{2a_*^2 M^3}{r_0^3} - \delta_6, \quad w_{00} = \frac{2a_* M^2}{r_0^2}. \end{aligned} \tag{4.7}$$

Notice that M is the black hole mass, r_0 is the radius of the black hole's event horizon and $a_* = \frac{J}{M^2}$ is the dimensionless black hole spin parameter where J is the black hole spin.

The deformation parameters $\{\delta_1, \delta_2, ..., \delta_6\}$ quantifies deviations from the Kerr metric. The physical interpretations of these parameters are listed below: δ_1 evaluates the deformations on g_{tt}; δ_2, δ_3 evaluate the rotational deformations; δ_4, δ_5 evaluate the deformations on g_{rr}; δ_6 evaluates the deformations on the event horizon. When all the deformation parameters are set to zero, the KRZ metric will reduce to the Kerr metric.

4.2 Further Calculation Process

The work we need to do now is to take the KRZ metric as the effective metric, calculate the effective one-body waveform and finally obtain the constraints on deformation parameters when the spins of the two black hole are under consideration.

For simplicity, we only take δ_1 into consideration and set all other deformation parameters to be zero. We study the inspiral phase of the gravitational waves emitted by the black hole binary systems, so the equatorial geodesics conditions $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ are imposed. Since the KRZ metric is the effective metric, M in previous formula is then equal to the total mass of the binary system $M = m_1 + m_2$. Assuming two black holes have aligned spins and setting $J = S_{eff}$, we can just follow the analysis proposed in the reference[24]. Due to the limit of time for this thesis, we only present what to do here and leave the calculation for future works.

Chapter 5
Discussion and Conclusions

This graduation thesis sketches out the acquired knowledge for studying gravitational waves and proposes a feasible calculation strategy to include the black hole spin into the gravitational wave analysis. We review the PN and EOB formalism which is fundamental tools in gravitational wave analysis. Then through summarizing people's previous works on spins, we find out the exact calculation process for the effective spin. Finally, we figure out how to develop a more advanced model for gravitational tests of the Kerr metric with binary black holes. The further work is to do the calculations. When the calculation is done, the model for gravitational wave tests can be more accurate.

Here, I want to discuss a little bit about the deformation parameters. The original goal of this thesis is to figure out the effective deformation parameters of the black hole binary systems through the effective one-body formalism, which shall follow the same calculation process of the effective spin in our initial envision.

However, the exact form of the effective spin is obtained through quantum field theory of gravity which we do not know much about. Resorting back to the effective one-body approach also seems not very wise since there is no transformation function for the metric like that for energy, turning them into an effective one from originally separated twos. The effective metric is assumed at start. Even though we could have performed the effective one-body metric transformation on the binary system, we would not obtain the exact form of the transformation relation since the QFT method is indispensable in the effective spin calculation. The EOB formalism only contributes partially.

In conclusion, including the black hole spin into gravitational testing model is feasible for us now. As for the deformation parameters, the analysis may need more new methods since they correspond to higher orders of the multipole moments of the black hole source[4].

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